Analysis of characteristic roots induced by a delayed feedback control for discrete time systems

Rinko Miyazaki[a,b], Dohan Kim[c] and Jong Son Shin[a]
[a]Shizuoka University, [b]Osaka University, [c]Seoul National University

1 Introduction and preliminaries

1.1 Introduction

The delayed feedback control (briefly, DFC) is proposed by Pyragas [7] as a method of chaos controls in continuous time systems. In this paper we consider a DFC for the discrete time system

$$x(n+1) = f(x(n)), \tag{E}$$

where $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is differentiable and $\mathbb{R}^d$ is the d-dimensional Euclidean space. For signals to stabilize an unstable periodic orbit (briefly, UPO) with period ω of Equation (E), there exist some types of DFC signals (for example. [13]). The signal given by

$$u(n) = K(x(n-\omega) - x(n)), \text{ (Pyragas type)}$$

is well-known as the typical signal of the DFC for continuous time systems (cf. [3,5,8,10–13]), where $d \times d$ real constant matrix K is the so-called *feedback gain*. Besides, Buchner and Żebrowski [1] consider the signals formulated as

$$u(n) = K(x(n-\omega) - f(x(n))), \quad \text{(Echo type)},$$

which is called DFC signal of "echo type". Ohta, Takahashi and Miyazaki [6] compare the validity of these two signals and suggest that Echo type is more effective than Pyragas type for one dimensional case. In the case where $K = kE$ (E is the $d \times d$ identity matrix), Miyazaki, Naito and Shin (cf. [4]) define a map which gives a relationship between characteristic multipliers of the original system (E) and that of the system with Echo type signals. Such a map is called "C-map".

The aim of this paper is to establish the general criteria on the stability of UPO for the DFC signals of Echo type by using C-map for the case where the gain matrix is $K = kE$ and the period of UPO is 2. More details, we determine the region of the characteristic multipliers of the target UPO of the original system (E) and the best range of feedback gain k so that the DFC signals of Echo type successfully stabilize the target UPO.

In order to describe our main results, we consider the first variational equation of Equation (E) around the target UPO,

$$x(n+1) = A(n)x(n), \tag{1.1}$$

and that of the system with DFC of Echo type signals,

$$y(n+1) = A(n)y(n) + K(y(n-\omega+1) - A(n)y(n)), \tag{1.2}$$

where $A(n)$, $n \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, is the Jacobian matrix of f evaluated at the target UPO and so that it is a periodic matrix function with period ω.

The paper is organized as follows. In Section 2, we give several lemmas on the equation deduced from the C-map theorem. In Section 3, we establish the general criteria on the stability of the solutions with period 2 for Equations (1.2), see Theorem 3.4. In particular, we determine the best range of feedback gain k so that the (unstable) periodic orbit is stabilized.

1.2 Preliminaries

In this subsection we give give a relationship between the characteristic multipliers for ordinary periodic linear difference equations (1.1) and delay periodic linear difference equations (1.2). Let $\mathbb{C}$ be the set of all complex numbers and $\mathbb{C}^d$ the d-dimensional complex Euclidean space. Let $L : X \to X$ be a bounded linear operator, where X is a Banach space with $\dim X < \infty$. We denote by $\sigma(L)$ the set of eigenvalues of L. Set $\sigma_{\mathbb{R}}(L) = \{\xi \in \mathbb{R} : \xi \in \sigma(L)\}$. Let $\mathbb{Z}_p = \{p, p+1, \cdots\}$, $p \in \mathbb{Z}$. Set $\mathbb{N} = \mathbb{Z}_1$ and $\mathbb{N}_0 := \mathbb{Z}_0$. Let $\mathcal{C}_{\omega-1}$ be the set of all maps from $\mathbb{Z}^0_{-\omega+1}$ into $\mathbb{C}^d$, which is a Banach space equipped with a sup norm $|\varphi|_{\mathcal{C}_{\omega-1}} = \sup_{s \in \mathbb{Z}^0_{-\omega+1}} |\varphi(s)|$.

We assume that the solution of Equation (1.1) through $(m, x^0) \in \mathbb{Z} \times \mathbb{C}^d$ and the solution of Equation (1.2) through $(m, \varphi) \in \mathbb{Z} \times \mathcal{C}_{\omega-1}$ exists uniquely. Then we denote by $T(n,m) : \mathbb{C}^d \to \mathbb{C}^d$ and $U(n,m) : \mathcal{C}_{\omega-1} \to \mathcal{C}_{\omega-1}$, respectively, the solution operators of Equation (1.1) and Equation (1.2). Set

$$T(n) = T(n+\omega, n), \text{ and } U(n) = U(n+\omega, n),$$

which are called the periodic operator. We note that

$$\sigma(T(n)) = \sigma(T(0)), \text{ and } \sigma(U(n)) = \sigma(U(0)).$$

Hereafter we assume that

(A) : $A(n)$ is nonsingular for all $n \in \mathbb{Z}$.

(C) : $KA(n) = A(n)K$, $(n \in \mathbb{Z})$;

(K) : (K-1) $\quad 0 \notin \sigma(K)$;

(K-2) $\quad 1 \notin \sigma(K)$;

(K-3) $\quad \sigma(U(0)) \cap \sigma(K) = \emptyset$.

Then we note that $0 \notin \sigma(T(0))$ and $0 \notin \sigma(U(0))$.

Now, we give a relationship between the characteristic multipliers of Equations (1.1) and (1.2). Set

$$g(y,z) = \left(\frac{z-y}{(1-y)z}\right)^{\omega}, \ (y \neq 1, y \neq z, z \neq 0).$$

A function $zg(k,z)$ is called the *characteristic multiplier function* (briefly, C-map) for Equation (1.2) with $K = kE$, where E is an identity matrix.

Theorem 1.1. (C-map Theorem) *Assume that* $K = kE$, $(0 < |k| < 1)$. *Then* $\nu \in \sigma(U(0))$ *if and only if* $\mu := \nu g(k,\nu) \in \sigma(T(0))$.

Refer to [4] for more details of this subsection.

2 Some of lemmas

In this section we consider the C-map $\mu = \nu g(k,\nu)$, that is,

$$\mu = \nu \left(\frac{\nu - k}{(1-k)\nu}\right)^{\omega}, \ (\mu, \nu \in \mathbb{C}), \tag{2.1}$$

where $\omega \in \mathbb{N}$ and $|k| < 1$. Moreover, setting $\nu = |\nu|e^{i\theta}, |\mu| \neq 0$ and taking the absolute value in the both sides of (2.1), we have

$$|\mu| = |\nu| \left(\frac{|\nu - k|}{(1-k)|\nu|}\right)^{\omega}. \tag{2.2}$$

The principal value, denoted by Arg μ, of the argument $\arg \mu$ of complex number $\mu = |\mu|e^{i\phi}$, is that unique value ϕ such that $-\pi < \phi \leq \pi$.

2.1 Lemmas on Equation (2.1)

In this subsection we will state characterize Equation (2.1). Let us define two functions

$$C_{\omega,k}(\nu) = \frac{(\nu - k)^{\omega}}{(1-k)^{\omega}\nu^{\omega-1}}, \ (\nu \in \mathbb{C})$$

and

$$B_{\omega,k}(\theta) = C_{\omega,k}(e^{i\theta}) = \frac{(e^{i\theta} - k)^{\omega}}{(1-k)^{\omega}e^{i(\omega-1)\theta}}, \ (\theta \in \mathbb{R}). \tag{2.3}$$

Denote by $n(\gamma, C_{\omega,k})$ the winding number of $C_{\omega,k}(\nu)$, when ν rotates along the unit cycle γ in the positive direction.

Lemma 2.1. *Let* $k \neq 1$. *Then the following statements hold.*

1) $B_{\omega,k}(2n\pi + \theta) = B_{\omega,k}(\theta)$ and $B_{\omega,k}(\theta) = \overline{B_{\omega,k}(-\theta)}$, $(n \in \mathbb{Z}, \theta \in (-\pi, \pi])$.
2) $B_{\omega,k}(0) \in \mathbb{R}$ *and* $B_{\omega,k}(\pi) = B_{\omega,k}(-\pi) \in \mathbb{R}$.
3) $C_{\omega,k}(\nu) = \overline{C_{\omega,k}(\overline{\nu})}$.
4) $n(\gamma, C_{\omega,k}) = 1$.

Proof. 1), 2) and 3) are obvious. 4) By argument principle, we have

$$\frac{1}{2\pi i}\int_{\gamma} \frac{C'_{\omega,k}(\nu)}{C_{\omega,k}(\nu)} d\nu = n(\gamma, C_{\omega,k}) = \omega - (\omega - 1) = 1$$

as required. $\square$

Note that there exist some intersection points of $B_{\omega,k}(\theta)$ in the real line by using the assertion 1) and 2) in Lemma 2.1.

Lemma 2.2. *Let $|k| < 1 (k \neq 0)$ and $\nu = e^{i\theta}, \theta \in (-\pi, \pi]$ in (2.1). Define $\beta := \beta(k, \theta)$ as*

$$\tan\beta = \frac{k\sin\theta}{1 - k\cos\theta}, \ \ |\beta| < \frac{\pi}{2}. \tag{2.4}$$

1) *If $\theta \neq 0, \pi$, then $\beta \neq 0$ and*

$$\begin{cases} k = \dfrac{\sin\beta}{\sin(\beta+\theta)}, \ \ |\sin(\beta+\theta)| > |\sin\beta| > 0 \\ \mu = \left\{\frac{\cos\frac{\theta}{2}}{\cos(\beta+\frac{\theta}{2})}\right\}^{\omega} e^{i\varphi}, \ \ 0 < |\mu| < \infty \\ \varphi = \omega\beta + \theta, \ \ 2\beta + \theta \neq \pm\pi. \end{cases} \tag{2.5}$$

2) *If $\theta = 0$, then $\beta = 0, \operatorname{Arg}\,\mu = 0$ and $\mu = 1$.*
3) *If $\theta = \pi$, then $\beta = 0, \operatorname{Arg}\,\mu = \pi$ and $\mu = -\left(\frac{1+k}{1-k}\right)^{\omega}$.*

Proof. Consider the function (2.3), that is,

$$\mu = B_{\omega,k}(\theta) = \frac{(e^{i\theta} - k)^{\omega}}{(1-k)^{\omega} e^{i(\omega-1)\theta}}, \ \ \theta \in (-\pi, \pi].$$

If $\theta = 0$, then it follows from (2.3) and (2.4) that $\beta = 0, \mu = 1$ and $\operatorname{Arg}\,\mu = 0$ hold. Similarly, if $\theta = \pi$, then $\beta = 0, \mu = -\left(\frac{1+k}{1-k}\right)^{\omega}$ and $\operatorname{Arg}\,\mu = \pi$.

Assume that $\theta \neq 0, \pi$. Then $\beta \neq 0$. Clearly, we have

$$\begin{aligned} \mu(1-k)^{\omega} &= (e^{i\theta} - k)^{\omega} e^{-i(\omega-1)\theta} \\ &= (e^{i\theta} - k)^{\omega} (e^{-i\theta})^{\omega} e^{i\theta} \\ &= e^{i\theta}\left(1 - ke^{-i\theta}\right)^{\omega}. \end{aligned} \tag{2.6}$$

Note that

$$(1 - k\cos\theta)^2 + (k\sin\theta)^2 = 1 - 2k\cos\theta + k^2 > 0.$$

Since

$$1 - ke^{-i\theta} = (1 - k\cos\theta) + ik\sin\theta = \sqrt{1 - 2k\cos\theta + k^2}\, e^{i\beta}, \tag{2.7}$$

we have

$$\left(1 - ke^{-i\theta}\right)^{\omega} = (1 - 2k\cos\theta + k^2)^{\frac{\omega}{2}} e^{i\omega\beta}.$$

Hence,

$$|\mu||1-k|^{\omega} = |1 - ke^{-i\theta}|^{\omega} = (1 - 2k\cos\theta + k^2)^{\frac{\omega}{2}} > 0.$$

By the definition of $\tan\beta$ we have

$$\frac{\sin\beta}{\cos\beta} = \frac{k\sin\theta}{1 - k\cos\theta},$$

and hence, $k\sin(\beta+\theta)=\sin\beta$ and $\sin\beta\neq 0$. Thus

$$k=\frac{\sin\beta}{\sin(\beta+\theta)}\ \text{and}\ |\sin(\beta+\theta)|>|\sin\beta|>0.$$

Since

$$1-k=\frac{\sin(\beta+\theta)-\sin\beta}{\sin(\beta+\theta)}=\frac{2\cos(\beta+\frac{\theta}{2})\sin\frac{\theta}{2}}{\sin(\beta+\theta)}, \tag{2.8}$$

we have $\cos(\beta+\frac{\theta}{2})\neq 0$, that is, $2\beta+\theta\neq\pm\pi$. Moreover, since (2.8) and

$$\begin{aligned}1-ke^{-i\theta}&=(1-k\cos\theta)+ik\sin\theta\\&=\frac{\sin\theta}{\sin(\beta+\theta)}[\cos\beta+i\sin\beta]\\&=\frac{\sin\theta}{\sin(\beta+\theta)}e^{i\beta},\end{aligned}$$

we obtain

$$\begin{aligned}\mu&=\left(\frac{1-ke^{-i\theta}}{1-k}\right)^{\omega}e^{i\theta}\\&=\left(\frac{\sin\theta}{\sin(\beta+\theta)}\right)^{\omega}\left(\frac{\sin(\beta+\theta)}{2\cos\left(\beta+\frac{\theta}{2}\right)\sin\frac{\theta}{2}}\right)^{\omega}e^{i(\omega\beta+\theta)}\\&=\left[\frac{\cos\frac{\theta}{2}}{\cos\left(\beta+\frac{\theta}{2}\right)}\right]^{\omega}e^{i\varphi},\end{aligned}$$

as required. $\square$

Remark 2.3. (i) Notice that there is an $l\in\mathbb{Z}$ such that

$$\varphi_0:=\operatorname{Arg}\mu=\varphi+2l\pi$$

when $\theta\neq 0$.

(ii) The statement $2\beta+\theta\neq\pm\pi$ in 2) of Lemma 2.2 says that if $\omega=2$, then $\varphi\neq\pm\pi$.

(iii) It follows from Lemma 2.2 that $0<|B_{\omega,k}(\theta)|<\infty$ for all $\theta\in(-\pi,\pi]$.

Corollary 2.4. *In Lemma* 2.2 *we obtain*

$$\mu(1-k)^{\omega}=(1-2k\cos\theta+k^2)^{\frac{\omega}{2}}e^{i\varphi},\ 1-2k\cos\theta+k^2>0,$$

and

$$|\varphi|<\left(\frac{\omega}{2}+1\right)\pi. \tag{2.9}$$

In particular, if $\omega=2$, then

$$\mu(1-k)^2=(1-2k\cos\theta+k^2)e^{i\varphi},\ 1-2k\cos\theta+k^2>0,\ \text{and}\ |\varphi|<2\pi.$$

Proof. The proof follows from (2.6), (2.7) and Lemma 2.2. Moreover, we obtain

$$\begin{aligned} |\varphi| &= |\omega\beta+\theta| \le \omega|\beta|+|\theta| \\ &< \left(\frac{\omega}{2}+1\right)\pi. \end{aligned}$$

This proves the corollary. □

The function (2.3) with $\omega = 2$ is written as

$$\mu = B_{2,k}(\theta) = \frac{(e^{i\theta}-k)^2}{(1-k)^2 e^{i\theta}},\ (\theta \in (-\pi,\pi]), \tag{2.10}$$

while (2.10) is refined in the following lemma.

Lemma 2.5. *Let $\omega = 2$ in Lemma 2.2. Then*

$$\mu = \frac{(k+1)^2}{k^2+2k\cos\varphi+1}e^{i\varphi}.\ \varphi = 2\beta+\theta \tag{2.11}$$

and

$$\cos\theta = |\mu|(\cos\varphi+1)-1\ (\theta \in (-\pi,\pi]). \tag{2.12}$$

Proof. If $\theta \neq 0, \pi$, from (2.5), we have

$$|\mu| = \frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\varphi}{2}} = \frac{2\cos^2\frac{\theta}{2}}{2\cos^2\frac{\varphi}{2}} = \frac{\cos\theta+1}{\cos\varphi+1}$$

or equivalently

$$\cos\theta = |\mu|(\cos\varphi+1)-1\ (\theta \neq 0,\pi).$$

If $\theta = 0$, then $\mu = B_{2,k}(0) = 1$. Taking $\varphi = 0$, (2.11) and (2.12) hold for $\theta = 0$. Since $|\mu| \neq 0$, we have that if $\theta = \pi$, then $\mu = B_{2,k}(\pi) = -\left(\frac{1+k}{1-k}\right)^2$. Taking $\varphi = \pm\pi$, (2.11) and (2.12) hold for $\theta = \pi$. Therefore (2.11) and (2.12) hold for $\theta \in (-\pi,\pi]$.

By Corollary 2.4 with $\omega = 2$, we have, in view of (2.12),

$$|\mu|(1-k)^2 = 1-2k\cos\theta+k^2 = 1-2k[|\mu|(\cos\varphi+1)-1]+k^2,$$

that is,

$$|\mu|(1+2k\cos\varphi+k^2) = (k+1)^2.$$

Since

$$k^2+2k\cos\varphi+1 = (1+k\cos\varphi)^2+k^2\sin^2\varphi > 0,$$

we obtain

$$|\mu| = \frac{(k+1)^2}{k^2+2k\cos\varphi+1},$$

as required. □

Corollary 2.6. *Let $\omega = 2$ in Lemma 2.2. Then the following results hold.*

1) $\varphi = 0$ *if and only if* $\theta = 0$*. In this case,* $\mu = 1$*.*

2) $\varphi = \pm\pi$ *if and only if* $\theta = \pi$*. In this case,* $\mu = -\left(\frac{1+k}{1-k}\right)^2$*.*

Proof. Let $\varphi = 0$. By Lemma 2.5 we get $\mu = 1$ and $1 + \cos\theta = 2|\mu|$. Thus $\cos\theta = 1$, i.e., $\theta = 0$. Let $\varphi = \pm\pi$. Then we have $\mu = -\left(\frac{1+k}{1-k}\right)^2$ by using (2.11). By Corollary 2.4 we have

$$\mu(1-k)^2 = -(1 - 2k\cos\theta + k^2),$$

i.e., $-(1+k)^2 = -(1 - 2k\cos\theta + k^2)$. Thus, $\cos\theta = -1$, i.e., $\theta = \pi$.

Conversely, if $\theta = 0$, or π, then the assertions follow from Lemma 2.5. $\square$

Proposition 2.7. *Let $|k| < 1 (k \neq 0)$ and let $\omega = 2$ in (2.1). Then $B_{2,k}(\theta) \in \mathbb{R}$ if and only if $\theta = 0, \pi$.*

Proof. Set $\mu = |\mu|e^{i\varphi} = B_{2,k}(\theta)$. It follows from Corollary 2.4 and Lemma 2.2 that

$$\mu(1-k)^\omega = (1 - 2k\cos\theta + k^2)e^{i\varphi}, \quad \varphi = \omega\beta + \theta.$$

Thus we have that $B_{2,k}(\theta) \in \mathbb{R}$ if and only if $\sin\varphi = 0$. Since $|\varphi| < \left(\frac{\omega}{2} + 1\right)\pi$, we obtain that $\sin\varphi = 0$ if and only if $\varphi = 2\beta + \theta = m\pi$, $|m| < \frac{\omega}{2} + 1 = 2$, that is, $m = 0, \pm 1$. Therefore $\varphi = 0, \pm\pi$. The result of the proposition follows from Corollary 2.6. $\square$

The curve $\partial\mathcal{D}^\alpha_{\omega,k} : \eta = B_{\omega,k}(\theta)$, $-\alpha < \theta \leq \alpha$, for some $0 < \alpha \leq \pi$ with $B_{\omega,k}(\alpha) = B_{\omega,k}(-\alpha)$, is closed. We denote by $int\mathcal{D}^\alpha_{\omega,k}$ and $ext\mathcal{D}^\alpha_{\omega,k}$ its interior and its exterior, respectively. Set $\mathcal{D}^\alpha_{\omega,k} = \partial\mathcal{D}^\alpha_{\omega,k} \cup int\mathcal{D}^\alpha_{\omega,k}$.

In particular, if the origin belongs to $int\mathcal{D}^\alpha_{\omega,k}$, then we denote by $\partial\mathcal{D}^\alpha_{\omega,k}(0)$ and $\mathcal{D}^\alpha_{\omega,k}(0)$ those.

Let $\mu = |\mu|e^{i\varphi} \in \mathbb{C}, \mu \neq 0$. We denote by L_μ the half line connecting the point μ from the origin. If $\partial\mathcal{D}^\alpha_{\omega,k} \cap L_\mu$ is unique, then there exists a unique $\theta_\mu \in (-\alpha, \alpha]$ such that $\eta_\mu = B_{\omega,k}(\theta_\mu)$. Hereafter, this argument θ_μ is called the argument associated with $(\mu, \partial\mathcal{D}^\alpha_{\omega,k}(0))$.

Definition 2.8. The closed curve $\partial\mathcal{D}^\alpha_{\omega,k}$ is called to be a strong star-shaped curve if it has the following properties :

(i) $\partial\mathcal{D}^\alpha_{\omega,k}$ is a simple closed curve.

(ii) $0 \in int\mathcal{D}^\alpha_{\omega,k}$, i.e., $\mathcal{D}^\alpha_{\omega,k} = \mathcal{D}^\alpha_{\omega,k}(0)$.

(iii) $\partial\mathcal{D}^\alpha_{\omega,k} \cap L_\mu$ is unique for every $\mu \in \mathbb{C}$.

Lemma 2.9. *Let $\omega = 2$, and $|k| < 1$. Then the closed curve $\partial\mathcal{D}^\pi_{2,k}$ is a strong star-shaped curve.*

Proof. Clearly, $\partial\mathcal{D}^\pi_{2,k}$ is a closed curve. The equation $\partial\mathcal{D}^\pi_{2,k} = \partial\mathcal{D}^\pi_{2,k}(0)$ follows from Lemma 2.1, i.e., $n(\gamma, C_{2,k}) = 1$. Now, we claim that $\partial\mathcal{D}^\pi_{2,k}$ is a simple closed curve. Since $\partial\mathcal{D}^\pi_{2,k}$ is continuous in $\theta \in (-\pi, \pi]$, we have to prove that $\partial\mathcal{D}^\pi_{2,k}$ is bijective, Without loss of generality, we assume that there exist $\theta_1, \theta_2, \theta_1 \neq \theta_2$ such that $\eta = |\eta|e^{i\varphi} = B_{2,k}(\theta_1) = B_{2,k}(\theta_2), \eta \in \mathbb{C} \setminus \mathbb{R}$. Then we have, by (2.12), that $\cos\theta_1 = \cos\theta_2 = |\eta|(\cos\varphi + 1) - 1$ holds. Namely, $\cos\theta_1 = \cos\theta_2$, and hence, $\theta_2 = -\theta_1$. Thus $B_{2,k}(\theta_2) = B_{2,k}(-\theta_1)$. On the othe hand, we have $\eta = B_{2,k}(\theta_1) = \overline{B_{2,k}(-\theta_1)}$ by Lemma 2.1. Therefore, $B_{2,k}(-\theta_1) = \overline{B_{2,k}(-\theta_1)}$. This means $\eta \in \mathbb{R}$, which is a contradiction. Hence $\theta_1 = \theta_2$, i.e., $\partial\mathcal{D}^{\pi}_{2,k}$ is bijective. Next, we will show the uniqueness of $\partial\mathcal{D}^{\pi}_{2,k}(0) \cap L_\mu$. Let $\mu = |\mu|e^{i\varphi}$. Assume that $\partial\mathcal{D}^{\pi}_{2,k}(0) \cap L_\mu = \{\eta_1, \eta_2\}, \eta_1 \neq \eta_2$. Since $\eta_1 = |\eta_1|e^{i\varphi}, \eta_2 = |\eta_2|e^{i\varphi}$, Lemma 2.5 implies that $\eta_1 = \eta_2$ holds. Therefore, $\partial\mathcal{D}^{\pi}_{2,k}(0) \cap L_\mu = \{\eta_\mu\}$. ◻

2.2 Lemmas on Equation (2.2)

In this subsection we consider Equation (2.2). Equation (2.2) becomes

$$|\mu|^{\frac{1}{\omega}}(1-k)|\nu|^{\frac{\omega-1}{\omega}} = |\nu - k|.$$

Squaring both sides, we have

$$|\mu|^{\frac{2}{\omega}}(1-k)^2|\nu|^{\frac{2(\omega-1)}{\omega}} = |\nu - k|^2 = |\nu|^2 - 2k|\nu|\cos\theta + k^2.$$

In relation with this, we introduce a function as follows :

$$f_\omega(r; k, \theta, |\mu|) = |\mu|^{\frac{2}{\omega}}(1-k)^2 r^{\frac{2(\omega-1)}{\omega}} - r^2 + 2kr\cos\theta - k^2, \tag{2.13}$$

defined on $0 \leq r < \infty,\ |k| < 1,\ -\pi < \theta \leq \pi$.

In particular, if $r = 1$, then

$$\begin{aligned} f_\omega(1; k, \theta, |\mu|) &= |\mu|^{\frac{2}{\omega}}(1-k)^2 - 1 + 2k\cos\theta - k^2 \\ &= (|\mu|^{\frac{2}{\omega}} - 1)k^2 - 2(|\mu|^{\frac{2}{\omega}} - \cos\theta)k + (|\mu|^{\frac{2}{\omega}} - 1). \end{aligned} \tag{2.14}$$

For Equation (2.1) we define a polynomial $P_{\omega,k}(\lambda; \mu)$ of λ, with degree ω, by

$$P_{\omega,k}(\lambda; \mu) = (\lambda - k)^\omega - \mu(1-k)^\omega \lambda^{\omega-1}. \tag{2.15}$$

Then Equation (2.1) is rewritten as $P_{\omega,k}(\lambda; \mu) = 0$. Clearly, Equation $P_{\omega,k}(\lambda; \mu) = 0$ has ω solutions, counted with multiplicity.

Lemma 2.10. *Let $\omega \in \mathbb{N}$, $k \in (-1, 1)$, $\mu \in \mathbb{C}$ and $\nu = |\nu|e^{i\theta_*}, |\mu| \neq 0$. Then $P_{\omega,k}(\nu; \mu) = 0$ if and only if $f_\omega(|\nu|; k, \theta_*, |\mu|) = 0$.*

Proof. Clearly, it is easy to see that $P_{\omega,k}(\nu; \mu) = 0$ if and only if

$$|\mu|^{\frac{2}{\omega}}(1-k)^2|\nu|^{\frac{2(\omega-1)}{\omega}} = |\nu|^2 - 2k|\nu|\cos\theta_* + k^2,$$

that is, $f_\omega(|\nu|; k, \theta_*, |\mu|) = 0$, and vice versa. ◻

Next, we will consider the conditions on k ($|k| < 1$) so that $f_\omega(1; k, \theta, |\mu|) < 0$. In particular, if $\theta = 0$, then

$$f_\omega(1; k, 0, |\mu|) = (|\mu|^{\frac{2}{\omega}} - 1)(k-1)^2 = (|\mu|^{\frac{1}{\omega}} - 1)(|\mu|^{\frac{1}{\omega}} + 1)(k-1)^2. \tag{2.16}$$

The following result is easily obtained from (2.16).

Lemma 2.11. *Let $\theta = 0$ in $f_\omega(1; k, \theta, |\mu|)$. Then the following statements hold for all $k \in (-1, 1)$:*
(1) $f_\omega(1; k, 0, |\mu|) < 0$ *if and only if* $|\mu| < 1$;
(2) $f_\omega(1; k, 0, |\mu|) > 0$ *if and only if* $|\mu| > 1$; *and*
(3) $f_\omega(1; k, 0, |\mu|) = 0$ *if and only if* $|\mu| = 1$.

Assume that $\theta \neq 0$. If $|\mu| = 1$, then

$$f_\omega(1; k, \theta, 1) = -2(1 - \cos\theta)k. \tag{2.17}$$

If $|\mu| \neq 1$, then (2.14) is rewritten as

$$f_\omega(1; k, \theta, |\mu|) = (|\mu|^{\frac{2}{\omega}} - 1)\left(k - \frac{|\mu|^{\frac{2}{\omega}} - \cos\theta}{|\mu|^{\frac{2}{\omega}} - 1}\right)^2 - \frac{D_\mu(\theta)}{|\mu|^{\frac{2}{\omega}} - 1}, \tag{2.18}$$

where

$$D_\mu(\theta) = \left(|\mu|^{\frac{2}{\omega}} - \cos\theta\right)^2 - \left(|\mu|^{\frac{2}{\omega}} - 1\right)^2.$$

By further calculation, we have

$$\begin{aligned} D_\mu(\theta) &= (1 - \cos\theta)\left\{2|\mu|^{\frac{2}{\omega}} - (1 + \cos\theta)\right\} \\ &= 4\sin^2\frac{\theta}{2}\left(|\mu|^{\frac{2}{\omega}} - \cos^2\frac{\theta}{2}\right) \\ &= 4\sin^2\frac{\theta}{2}\left(|\mu|^{\frac{1}{\omega}} + \cos\frac{\theta}{2}\right)\left(|\mu|^{\frac{1}{\omega}} - \cos\frac{\theta}{2}\right). \end{aligned} \tag{2.19}$$

We note that if $\theta \neq 0$, then the following statements hold :
(1) $D_\mu(\theta) > 0$ if and only if $|\mu| > \cos^\omega\frac{\theta}{2}$;
(2) $D_\mu(\theta) = 0$ if and only if $|\mu| = \cos^\omega\frac{\theta}{2}$; and
(3) $D_\mu(\theta) < 0$ if and only if $|\mu| < \cos^\omega\frac{\theta}{2}$.

In the case $D_\mu(\theta) > 0$, the quadratic equation $f(1; k, \theta, |\mu|) = 0$ of k has two real solutions :

$$k_\pm(\theta) := \frac{|\mu|^{\frac{2}{\omega}} - \cos\theta \pm \sqrt{D_\mu(\theta)}}{|\mu|^{\frac{2}{\omega}} - 1}.$$

Lemma 2.12. *Let $-\pi < \theta \leq \pi, \theta \neq 0$ and $|k| < 1$. Then the following statements hold :*
1) *Let $|\mu| > 1$. Then the inequalities $0 < k_-(\theta) < 1 < k_+(\theta)$ hold. Moreover, if $k_-(\theta) < k < 1$ then $f_\omega(1; k, \theta, |\mu|) < 0$; if $-1 < k < k_-(\theta)$ then $f_\omega(1; k, \theta, |\mu|) > 0$.*

2) *Let* $|\mu| < 1$.

(2-1) *In case* $|\mu| > \cos^\omega \frac{\theta}{2}$, *the inequalities* $k_+(\theta) < -1 < k_-(\theta) < 0$ *hold. Moreover, if* $k_-(\theta) < k < 1$ *then* $f_\omega(1; k, \theta, |\mu|) < 0$; *if* $-1 < k < k_-(\theta)$ *then* $f_\omega(1; k, \theta, |\mu|) > 0$.

(2-2) *In case* $|\mu| \leq \cos^\omega \frac{\theta}{2}$, $f_\omega(1; k, \theta, |\mu|) < 0$ *for all* $k \in (-1, 1)$.

3) *Let* $|\mu| = 1$. *Then if* $0 < k < 1$ *then* $f_\omega(1; k, \theta, |\mu|) < 0$; *if* $-1 < k < 0$ *then* $f_\omega(1; k, \theta, |\mu|) > 0$.

Proof. Set $f_\theta(k) = f_\omega(1; k, \theta, |\mu|)$.

1) Since $|\mu| > 1 \geq \cos^\omega \frac{\theta}{2}, \theta \neq 0$, we find that $D_\mu(\theta) > 0$. Then the quadratic equation $f_\theta(k) = 0$ of k has two real solutions $k_-(\theta)$ and $k_+(\theta)$. By noting the following facts that

$$f_\theta(0) = |\mu|^{\frac{2}{\omega}} - 1 > 0$$

and

$$f_\theta(1) = -2(1 - \cos\theta) < 0,$$

we can easily see that the assertions are true.

2) (2-1) By the assumptions $1 > |\mu| > \cos^\omega \frac{\theta}{2}$ and $\theta \neq 0$, we find that $D_\mu(\theta) > 0$. Then the quadratic equation $f_\theta(k) = 0$ of k has two real solutions $k_-(\theta)$ and $k_+(\theta)$. Noting that

$$f_\theta(0) = |\mu|^{\frac{2}{\omega}} - 1 < 0$$

and

$$\begin{aligned} f_\theta(-1) &= 4|\mu|^{\frac{2}{\omega}} - 2 - 2\cos\theta \\ &= 4\left(|\mu|^{\frac{2}{\omega}} - \cos^2\frac{\theta}{2}\right) \\ &= \left(|\mu|^{\frac{1}{\omega}} + \cos\frac{\theta}{2}\right)\left(|\mu|^{\frac{1}{\omega}} - \cos\frac{\theta}{2}\right) > 0, \end{aligned}$$

we can easily see that the assertions are true.

(2-2) If $|\mu| = \cos^\omega \frac{\theta}{2}$, then $|\mu|^{\frac{2}{\omega}} = \cos^2\frac{\theta}{2} = \frac{1}{2}(\cos\theta + 1)$ and we have

$$\begin{aligned} f_\theta(k) &= (|\mu|^{\frac{2}{\omega}} - 1)k^2 - 2\{|\mu|^{\frac{2}{\omega}} - (2|\mu|^{\frac{2}{\omega}} - 1)\}k + |\mu|^{\frac{2}{\omega}} - 1 \\ &= (|\mu|^{\frac{2}{\omega}} - 1)(k+1)^2 < 0. \end{aligned}$$

If $|\mu| < \cos^\omega \frac{\theta}{2}$, then we find that $D_\mu(\theta) < 0$, so that $f_\theta(k) < 0$ from (2.18).

3) If $|\mu| = 1$, $f_\theta(k)$ is given by (2.17). Then we can easily see that the assertion 3) is true. $\square$

3 Stabilization by DFC : $\omega = 2$

In this section we will analyze the stability of DFC for the case $\omega = 2$.

Proposition 3.1. *Assume that $\omega \geq 2$ and $K = kE$. If $|k| > 1$, then there exists $\nu \in \sigma(U(0))$ such that $|\nu| > 1$.*

Proof. Let $P_{\omega,k}(\lambda;\mu)$ be the polinomial defined by (2.15). Then $P_{\omega,k}(\lambda;\mu) = 0$ and $P_{\omega,k}(0;\mu) = (-k)^{\omega}$. On the other hand, since

$$P_{\omega,k}(\lambda;\mu) = (\lambda - \nu_1)(\lambda - \nu_2)\cdots(\lambda - \nu_{\omega}),$$

we obtain $P_{\omega,k}(0;\mu) = (-1)^{\omega}\nu_1\nu_2\cdots\nu_{\omega}$. Summing up these, we arrive at

$$\nu_1\nu_2\cdots\nu_{\omega} = (k)^{\omega}. \tag{3.1}$$

This shows that if $|k| > 1$, then there exists $\nu_i \in \sigma(U(0))$ such that $|\nu_i| > 1$ holds. □

Proposition 3.2. *Let $\omega = 2, |k| < 1$ and $\mu = |\mu|e^{i\varphi} \in \mathbb{C}$ be given and let θ_μ be the argument associated with $(\mu, \partial\mathcal{D}^{\pi}_{2,k}(0))$. Then the following statments are equivalent :*

1)
$$\mu \in int\mathcal{D}^{\pi}_{2,k}(0).$$

2)
$$|\mu| < |\eta_\mu| := |B_{2,k}(\theta_\mu)| = \frac{k^2 - 2k\cos\theta_\mu + 1}{(1-k)^2}.$$

3)
$$|\mu| < \frac{(k+1)^2}{k^2 + 2k\cos\varphi + 1}.$$

4)
$$f_2(1; k, \theta_\mu, |\mu|) < 0.$$

5)
$$g_f(1; k, \varphi, |\mu|) < 0,$$

where

$$g_f(1; k, \varphi, |\mu|) = |\mu|(1-k)^2 - 1 + 2k(|\eta_\mu|(\cos\varphi + 1) - 1) - k^2, \quad \eta_\mu = B_{2,k}(\theta_\mu).$$

Proof. The assertions 1) and 2) are equivalnt. It follows from Lemma 2.9 that if $\mu = |\mu|e^{i\varphi} \in int\mathcal{D}^{\pi}_{2,k}(0)$, then $|\mu| < |B_{2,k}(\theta_\mu)|$. This means the assertion 2), and vice versa. The equivalence of 2) and 4) is shown as follows. Assuming that 2) holds, we have $|\mu|(1-k)^2 < |e^{i\theta_\mu} - k|^2$, and hence,

$$|\mu|(1-k)^2 < |e^{i\theta_\mu} - k|^2.$$

Since
$$|e^{i\theta_\mu} - k|^2 = 1 - 2k\cos\theta_\mu + k^2,$$
we can obtain $f_2(1; k, \theta_\mu, |\mu|) < 0$, and vice versa.

We derive 5) from 3).

$$\begin{aligned}
&g_f(1; k, \varphi, |\mu|) \\
&= -1 + |\mu|(1-k)^2 + 2k[|\eta_\mu|(\cos\varphi + 1) - 1] - k^2 \\
&< |\eta_\mu|(1-k)^2 + 2k|\eta_\mu|(\cos\varphi + 1) - (k+1)^2 \\
&= |\eta_\mu|(k^2 + 2k\cos\varphi + 1) - (k+1)^2 \\
&= (k+1)^2 - (k+1)^2 \\
&= 0.
\end{aligned}$$

The remainder follows from Lemma 2.9 and Lemma 2.5. □

Remark 3.3. In Proposition 3.2, if $\mu \in ext\mathcal{D}^{\pi}_{2,k}(0)$, then the inequality $<$ can be replaced by the inequality $>$.

Set
$$\sigma_{(\mu,K)}(U(0)) = \{\nu \in \mathbb{K} \cap \sigma(U(0)) : \mu = \nu\left(\frac{\nu - k}{(1-k)\nu}\right)^{\omega} \in \sigma(T(0))\}.$$

Now, we are in a position to atate the main theorem in this paper.

Theorem 3.4. *Let $\omega = 2$ and $K = kE$ and $|k| < 1, (k \neq 0)$. Let $\mu \in \sigma(T(0))$.*

1) If $\mu \in int\mathcal{D}^{\pi}_{2,k}(0)$, then $|\nu| < 1$ for all $\nu \in \sigma_{(\mu,C)}(U(0))$.

2) If $\mu \in ext\mathcal{D}^{\pi}_{2,k}(0)$, then there exists a $\nu \in \sigma_{(\mu,C)}(U(0))$ such that $|\nu| > 1$.

Proof. Since $\omega = 2$, we have that $B_{2,k}(\theta) \in \mathbb{R}$ if and only if $\theta = 0, \pi$ by Proposition 2.7. Let $\mu = |\mu|e^{i\varphi} \in \sigma(T(0)), \mu \neq 0$. Without loss of generality, we assume $0 < \varphi \leq \pi$. Then $\mu = C_{2,k}(\nu)$. Let $\nu = |\nu|e^{i\theta_\mu} \in \sigma_{(\mu,C)}(U(0))$. Then θ_μ is the argument associated with $(\mu, \partial\mathcal{D}^{\pi}_{2,k}(0))$.

1) It is sufficient to prove that $0 < |\nu| < 1$ holds, provided $\mu \in int\mathcal{D}^{\pi}_{2,k}(0)$. Taking $\omega = 2$ and $\theta = \theta_\mu$ in (2.13) we have
$$f(r) := f_2(r; k, \theta_\mu, |\mu|) = |\mu|(1-k)^2 r - r^2 + 2kr\cos\theta_\mu - k^2.$$

Then $f_2(|\nu|; k, \theta_\mu, |\mu|) = 0$ by Lemma 2.10. Moreover, by (2.12) the equation $\cos\theta_\mu = |\eta_\mu|(\cos\varphi + 1) - 1$ holds, and hence,
$$g_f(r) := g_f(r; k, \varphi, |\mu|) = -r^2 + |\mu|(1-k)^2 r + 2k[|\eta_\mu|(\cos\varphi + 1) - 1]r - k^2. \tag{3.2}$$

Then $g_f(0) = -k^2 < 0$ and $g_f(1) < 0$. Indeed, by Proposition 3.2 and (3.2), we have $g_f(1) < 0$ holds.

Then we prove that if $g_f(1) < 0$, then $g_f'(r) < 0$ for all $r \in [1, \infty)$.

$$\begin{aligned} g_f'(r) &= -2r + |\mu|(1-k)^2 + 2k[|\eta_\mu|(\cos\varphi + 1) - 1] \\ &= -2r + g_f(1) + 1 + k^2 \\ &= (1 + k^2 - 2r) + g_f(1) \\ &< g_f(1) < 0. \end{aligned}$$

Next, we take $r_0, 0 \le r_0 < 1$ so that

$$\max_{0 \le r \le 1} g_f(r) = g_f(r_0).$$

Then r_0 is a point of local maximum of $g_f(r)$. Now, we prove $g_f(r_0) \ge 0$. If $g_f(r_0) < 0$, then $g_f(r) < 0$ for all $r \ge 0$.

Thus the equation $f(r) = 0$ has no solution, but $f_\omega(|\nu|; k, \theta_\mu, |\mu|) = 0$ holds. This yeilds a contradiction. Thus $f(r_0) \ge 0$. Summing up these, we can obtain $0 < |\nu| < 1$ holds.

2) Since $\mu \in ext\mathcal{D}^{\pi}_{2,k}(0)$, it follows that

$$|\mu| > |\eta_\mu| = \frac{(k+1)^2}{k^2 + 2k\cos\varphi + 1} \tag{3.3}$$

holds. Then $g_f(1; k, \varphi, |\mu|) > 0$. Indeed, by (3.3) we have

$$\begin{aligned} & g_f(1; k, \varphi, |\mu|) \\ &= -1 + |\mu|(1-k)^2 + 2k[|\eta_\mu|(\cos\varphi + 1) - 1] - k^2 \\ &> |\eta_\mu|(1-k)^2 + 2k|\eta_\mu|(\cos\varphi + 1) - (k+1)^2 \\ &= |\eta_\mu|(k^2 + 2k\cos\varphi + 1) - (k+1)^2 \\ &= (k+1)^2 - (k+1)^2 \\ &= 0. \end{aligned}$$

Since $f(0) < 0$, by the theorem of intermediate value there exists an $r_0, 0 < r_0 < 1$, satisfying $f(r_0) = 0$. Another solution exists on $(1, \infty)$. This implies that there exists a $\nu \in \sigma_{(\mu,C)}(U(0))$ such that $|\nu| > 1$. $\square$

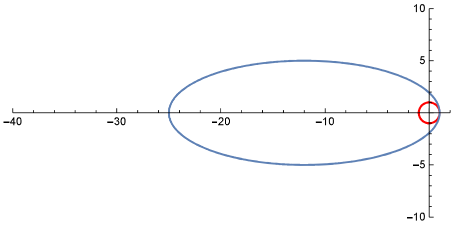

Figure 1: The region of $\mu \in \sigma(T(0))$ so that unstable periodic orbit is stabilized in the case where $\omega = 2,\ k = 2/3$.

Theorem 3.4 gives the region of $\mu \in \sigma(T(0))$ in which DFC with the gain K = kE is successful. Fig. 3 illustrate the region in the case where $\omega = 2$, $k = 2/3$. The blue curve shows $\partial\mathcal{D}^{\pi}_{\omega,\|}$ and if all $\mu \in \sigma(T(0))$ are in the interior of the curve, then the unstable periodic orbit is stabilized.

Based on Theorem 3.4, we bellow give the best range of feedback gain k so that the (unstable) periodic orbit is stabilized.

Corollary 3.5. *Let $\mu = |\mu|e^{i\varphi} \in \sigma_C(T(0))$ be as in Theorem 3.4. Set*

$$k_{\pm}(\varphi) = \frac{1 - |\mu|\cos\varphi \pm \sqrt{(|\mu|\cos\varphi - 1)^2 - (|\mu| - 1)^2}}{|\mu| - 1}.$$

1) If $1 < |\mu| < \frac{2}{\cos\varphi + 1}, |\mu| \neq 1$ and $k_-(\varphi) < k < k_+(\varphi)$, then $|\nu| < 1$ for all $\nu \in \sigma_{(\mu,C)}(U(0))$;

2) If $|\mu| < 1$ and $k < k_-(\varphi) < 0$ or $k_+(\varphi) < k < 1$, then $|\nu| < 1$ for all $\nu \in \sigma_{(\mu,C)}(U(0))$;

3) If $|\mu| = 1$ and $0 < k < 1$, then $|\nu| < 1$ for all $\nu \in \sigma_{(\mu,C)}(U(0))$; and

4) If $|\mu| > \frac{2}{\cos\varphi + 1}$ and $|k| < 1$, the there is a $\nu \in \sigma_{(\mu,C)}(U(0))$ such that $|\nu| > 1$.

Proof. Since $\mu \in int\mathcal{D}^{\pi}_{2,k}(0)$, the assertion 3) in Proposition 3.2 is rewritten as

$$h(k) := |\mu|(k^2 + 2k\cos\varphi + 1) - (k+1)^2 < 0$$

i.e.,

$$h(k) = (|\mu| - 1)k^2 + 2(|\mu|\cos\varphi - 1)k + (|\mu| - 1) < 0. \tag{3.4}$$

The solutions of Equation $h(k) = 0$ are expressed as

$$k_{\pm}(\varphi) = \frac{1 - |\mu|\cos\varphi \pm \sqrt{D_h}}{|\mu| - 1},$$

where

$$D_h = (|\mu|\cos\varphi - 1)^2 - (|\mu| - 1)^2.$$

Clearly,

$$D_h = |\mu|[|\mu|(\cos\varphi + 1) - 2](\cos\varphi - 1).$$

Therefore if $|\mu| \leq \frac{2}{\cos\varphi + 1}$, then $D_h \geq 0$.

1) Case $|\mu| > \frac{2}{\cos\varphi+1}$. Then $h(k) > 0$ for all k ($|k| < 1$).

2) Case $1 < |\mu| < \frac{2}{\cos\varphi+1}$. Then $h(k) < 0$ if $k_-(\varphi) < k < k_+(\varphi)$.

3) Case $|\mu| < 1$. Then $h(k) < 0$ if $k < k_-(\varphi) < 0$ or $k_+(\varphi) < k < 1$.

4) Case $|\mu| = 1$. Then $h(k) < 0$ if $0 < k < 1$.

Therefore the proof follows from Theorem 3.4. □

The following corollary is the case where $\mu \in \sigma(T(0))$ is a real number.

Corollary 3.6. *Let $\mu \in \sigma_R(T(0))$. Then the following statements hold.*

1) *If $\mu > 1$, then there exists $\nu \in \sigma_{(\mu,C)}(U(0))$ such that $\nu > 1$.*
2) *If $0 < \mu < 1$ and $|k| < 1, k \neq 0$, then $|\nu| < 1$ for all $\nu \in \sigma_{(\mu,C)}(U(0))$.*
3) *If $\mu < 0$ and*

$$\frac{\sqrt{-\mu}-1}{\sqrt{-\mu}+1} < k < 1,$$

then $|\nu| < 1$ for all $\nu \in \sigma_{(\mu,C)}(U(0))$.

Proof. The assertion 1) is obvious from the assertion 4) in Corollary 3.5. In order to prove the assertions 2) and 3), we apply the assertions 2) and 3) in Corollary 3.5.

Let $0 < \mu < 1$. Then $\varphi = 0$. Thus it follows from (3.4) that $h(k) = (\mu-1)(k+1)^2 < 0$ if and only if $0 < \mu < 1$ and $|k| < 1, k \neq 0$.

Let $\mu < 0$. Then $\varphi = \pi$. Thus (3.4) becomes

$$h(k) = (-\mu-1)k^2 + 2(\mu-1)k + (-\mu-1) < 0,$$

and hence, the solutions $k_\pm(\pi)$ of $h(k) = 0$ become

$$\begin{aligned} k_\pm(\pi) &= -\frac{-(\mu-1) \pm 2\sqrt{-\mu}}{\mu+1} \\ &= -\frac{(1-\mu) \pm 2\sqrt{-\mu}}{\mu+1} \\ &= -\frac{(\sqrt{-\mu} \pm 1)^2}{(1-\sqrt{-\mu})(1+\sqrt{-\mu})}. \end{aligned}$$

Hence

$$k_-(\pi) = \frac{\sqrt{-\mu}-1}{\sqrt{-\mu}+1} \quad \text{and} \quad k_+(\pi) = \frac{\sqrt{-\mu}+1}{\sqrt{-\mu}-1} > 1.$$

Hence, $h(k) < 0$ if and only if $k_-(\pi) < k < 1$. □

Remark 3.7. Another method for the proof of the assertion 3) in Corollary 3.6 is as follows. By Corollary 2.6 we have that $\mu \in int\mathcal{D}^\pi_{2,k}(0)$ if and only if

$$-\left(\frac{1+k}{1-k}\right)^2 < \mu < 1.$$

Thus if

$$\frac{\sqrt{-\mu}-1}{\sqrt{-\mu}+1} < k < 1,$$

then $|\nu| < 1$ for all $\nu \in \sigma_{(\mu,C)}(U(0))$.

The result of Corollary 3.6 coincides with one due to the Jury criterion (cf. [6]).

References

[1] T. Buchner and J. J. Żebrowski, Logistic map with a delayed feedback: Stability of a discrete time-delay control of chaos, *Phys. Rev. E*, **63**, (2000), 06210.

[2] S. Elaydi, *An Introduction to Difference Equations, 1st. Ed.*, Springer–Verlag, New York, 1996.

[3] R. Miyazaki, T. Naito, and J. S. Shin, Delayed feedback control by commutative gain matrices, *SIAM J. Math. Anal.*, **43**, (2011), 1122–1144.

[4] R. Miyazaki, T. Naito and S. J. Shin, Commutative gain matrices in delayed feedback controls, *RIMS-Kokyuroku*, **1702**, (2010), 51–64. (in Japanese)

[5] Ö. Morgül, On the stability of delayed feedback controllers, *Phys. Lett. A*, **314**, (2003), 278–285.

[6] T. Ohta, K. Takahashi and R. Miyazaki, On the stabilization method of periodic orbits in difference equations, *RIMS-Kokyuroku*, **1445**, (2005), 129–136. (in Japanese)

[7] K. Pyragas, Continuous control of chaos by self-controlling feedback, *Physics Letters A*, **170**, (1992), 421–428.

[8] K. Pyragas *et. al.*, Delayed feedback control of chaos, *Phil. Trans. Royal. Soc. A*, (2006), **364**, 2309–2334.

[9] T. Ushio and S. Yamamoto, Prediction-based control of chaos, *Phys. Lett. A*, **264**, (1999), 30–35.

[10] T. Ushio, Limitation of delayed feedback control in nonlinear discrete-time systems, *IEEE Trans. Circuits Syst. I*, **43**, (1996), 815–816.

[11] M. S. Vieira and A. J. Lichtenberg, Controlling chaos using nonlinear feedback with delay, *Phys. Rev. E*, **54**, (1996), 1200.

[12] J. Zhu and Y.-P. Tian, Necessary and sufficient conditions for stabilizability of discrete-time systems via delayed feedback control, *Physics Letters A*, **343**, (2005), 95–107.

[13] D. Yang and J. Zhou, Connections among several chaos feedback control approaches and chaotic vibration control of mechanical systems, *Commun Nonlinear Sci Numer Simulat*, **19**, (2014), 3954–3968.